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Duality in quantum Minkowski spacetime

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Abstract. The quantum Minkowski spacetime has real structure and this seems to be contradictive to the differential calculus in it. Dual differentiations are introduced to solve this problem. This duality can be extended to differential calculus in any C^*-algebra.

Non-commutative geometry is an active research topic in modern mathematics and also a new formulation for spaces and fields in modern physics [1–10]. The relativistic quantum field theory constructed in conventional Minkowski spacetime is perhaps not applicable when physics is studied at the sub-microscopic scale [11]. Therefore the quantum Minkowski spacetime based on the q-deformed Lorentz group, [11–18], which is essentially non-commutative, was suggested. The differential calculus in this space is important in order to formulate field theories, and many authors have studied it [12, 13, 17, 18]. However, there is still an ambiguity to be solved, i.e. the contradiction between the reality of the space and the differential in the space, the former of which is physically required and important.

A method of constructing differential calculus in quantum spaces has been suggested by Wess and Zumino [19]. They introduced Cartan's exterior differentiation d into the space, and the differential algebra A is then generated by 'coordinate' x^μ and its differential

$$\xi^\mu \equiv \mathrm{d}x^\mu \tag{1}$$

modulo some commutation relations. A is graded according to the order p in ξ^μ,

$$A = \bigoplus_{p=0}^{N} A_p \tag{2}$$

where N is a dimension of the space, and the operation d is a mapping in A,

$$\mathrm{d} : A_p \longrightarrow A_{p+1} \tag{3}$$

which satisfies the axioms

$$\begin{cases} \mathrm{d}^2 = 0 & \text{(Cartan rule)} \\ \mathrm{d}(fg) = (\mathrm{d}f)g + (-1)^{p(f)} f(\mathrm{d}g) & \text{(Leibniz rule)} \end{cases} \tag{4}$$

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where $p(f)$ is the order of the element f in the algebra. They also defined partial differentiation ∂_μ through

$$\mathrm{d} = \xi^\mu \partial_\mu \qquad \text{(summed over } \mu\text{)}. \tag{5}$$

Then the calculus is determined by the requirement of consistency, starting with the commutation relations for coordinate x^μ. Furthermore Carow-Watamura *et al* [20] pointed out that when a metric exists in a quantum space, the algebra becomes a Birman–Wenzl–Murakami algebra [21,22]. This is also the case of quantum Minkowski space. By means of the metric $g_{\mu\nu}$ the quadratic invariant centre of the algebra is

$$J = g_{\mu\nu} x^\mu x^\nu \tag{6}$$

and the subscript of partial differentiation can be transformed to a superscript,

$$\partial^\mu = g^{\mu\nu} \partial_\nu \tag{7}$$

where $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$. According to this method the differential calculus in quantum Minkowski space (in compact tensor form) is [17,18]

$$\begin{aligned}
&(\mathcal{P}_A)_{12} x_1 x_2 = 0 \\
&(\mathcal{P}_S)_{12} \xi_1 \xi_2 = (\mathcal{P}_1)_{12} \xi_1 \xi_2 = 0 \\
&x_1 \xi_2 = q \hat{\mathcal{R}}_{12} \xi_1 x_2 \\
&\partial_1 x_2 = (G^{-1})_{12} + q (\hat{\mathcal{R}}^{-1})_{12} x_1 \partial_2 \\
&\partial_1 \xi_2 = q^{-1} \hat{\mathcal{R}}_{12} \xi_1 \partial_2 \\
&(\mathcal{P}_A)_{12} \partial_1 \partial_2 = 0
\end{aligned} \tag{8}$$

where

$$x = (x^\mu) \qquad \xi = (\xi^\mu) \qquad \partial = (\partial^\mu) \qquad G^{-1} = (g^{\mu\nu})$$

$\mathcal{P}_i (i = 1, A, S)$ are the projective operators for singlet, antisymmetric and symmetric multiplets, respectively, and $\hat{\mathcal{R}}$ is the $\mathcal{R}$ matrix of the vector representation of quantum group $\mathrm{SL}_q(2, \mathbb{C})$ (q-deformed Lorentz group). With a proper choice of basis ($\mu = 0, +, 3, -$) the commutation relation for coordinate x^μ is

$$\begin{aligned}
&x^0 x^+ - x^+ x^0 = x^0 x^3 - x^3 x^0 = x^0 x^- - x^- x^0 = 0 \\
&q x^+ x^3 - q^{-1} x^3 x^+ = \omega x^0 x^+ \\
&q x^3 x^- - q^{-1} x^- x^3 = \omega x^0 x^- \\
&x^+ x^- - x^- x^+ = \omega (x^3 - x^0) x^3
\end{aligned} \tag{9}$$

where

$$\omega = q - q^{-1}$$

and the invariant metric is

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -q \\ 0 & 0 & -1 & 0 \\ 0 & -q^{-1} & 0 & 0 \end{pmatrix} \qquad (\mu, \nu = 0, +, 3, -). \tag{10}$$

This differential calculus is covariant under quantum Lorentz transformation

$$x'^{\mu} = L^{\mu}{}_{\nu} x^{\nu} \qquad \xi'^{\mu} = L^{\mu}{}_{\nu} \xi^{\nu} \qquad \partial'^{\mu} = L^{\mu}{}_{\nu} \partial^{\nu} \tag{11}$$

where the quantum Lorentz matrix $L^{\mu}{}_{\nu}$ satisfies the Yang–Baxter relation

$$\hat{\mathcal{R}}_{12} L_1 L_2 = L_1 L_2 \hat{\mathcal{R}}_{12}. \tag{12}$$

and the $\hat{\mathcal{R}}$ matrix itself satisfies Yang–Baxter equation

$$\hat{\mathcal{R}}_{12} \hat{\mathcal{R}}_{23} \hat{\mathcal{R}}_{12} = \hat{\mathcal{R}}_{23} \hat{\mathcal{R}}_{12} \hat{\mathcal{R}}_{23}. \tag{13}$$

Now the problem is that, in the quantum group $\mathrm{SL}_q(2,\mathbb{C})$ and its representation spaces regarded as a C^*-algebra, the $*$-conjugation has been introduced, which obeys the axioms

$$\begin{cases} (f^*)^* = f & \text{(idempotence)} \\ (fg)^* = g^* f^* & \text{(algebraic antihomomorphism)} \end{cases} \tag{14}$$

for any element f and g in the algebra, and quantum Minkowski space is a 'real' representation of the quantum group $\mathrm{SL}_q(2,\mathbb{C})$ with real q (see [12]). This means that there exists a relation for x^{μ} and its conjugation

$$(x^{\mu})^* = C^{\mu}{}_{\nu} x^{\nu} \tag{15}$$

where $C = (C^{\mu}{}_{\nu})$ is a real matrix invariant under a quantum Lorentz transformation. Explicitly the matrix C is

$$C^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & q^{-1} \\ 0 & 0 & 1 & 0 \\ 0 & q & 0 & 0 \end{pmatrix} \qquad (\mu, \nu = 0, +, 3, -). \tag{16}$$

However it is not difficult to see that the operations d and $*$ must be non-commutative, i.e. $(\mathrm{d}f)^* \neq \mathrm{d}(f^*)$, because the Leibniz rule for exterior differentiation and the algebraic antihomomorphism for $*$-conjugation contradict their commutivity in the case of general non-commutative algebra, and also the commutation relations (8) are not consistent if $(\xi^{\mu})^* = C^{\mu}{}_{\nu}\xi^{\nu}$. Therefore it is not possible to extend $*$-conjugation from the algebra A_0 to the graded algebra A, if one wants to retain all the fundamental axioms.

To solve this problem we observed that the form of the Leibniz rule is dependent on the direction in which the differential applies. Quite parallel to (4) we can also introduce right differential $\overleftarrow{\mathrm{d}}$ which obeys the axioms

$$\overleftarrow{\mathrm{d}}^2 = 0 \qquad (fg)\overleftarrow{\mathrm{d}} = f(g\overleftarrow{\mathrm{d}}) + (-1)^{p(g)}(f\overleftarrow{\mathrm{d}})g \tag{17}$$

and the new element

$$\eta^\mu \equiv x^\mu \overleftarrow{\mathrm{d}} \tag{18}$$

then x^μ and η^μ generate another graded algebra A'. We also define right partial differentiation $\overleftarrow{\partial}_\mu$ through

$$\overleftarrow{\mathrm{d}} = \overleftarrow{\partial}_\mu \, \eta^\mu \tag{19}$$

and raise its subscript by

$$\overleftarrow{\partial}^\mu = \overleftarrow{\partial}_\nu \, g^{\nu\mu} \tag{20}$$

η^μ and $\overleftarrow{\partial}^\mu$ also transform under quantum Lorentz transformation as

$$\eta'^\mu = L^\mu{}_\nu \eta^\nu \qquad \overleftarrow{\partial}'^\mu = L^\mu{}_\nu \, \overleftarrow{\partial}^\nu . \tag{21}$$

Then we have a right differential calculus in the algebra A', comparable with equation (8),

$$\begin{aligned}
&(\mathcal{P}_S)_{12}\eta_1\eta_2 = (\mathcal{P}_1)_{12}\eta_1\eta_2 = 0 \\
&\eta_1 x_2 = q\hat{\mathcal{R}}_{12} x_1 \eta_2 \\
&x_1 \, \overleftarrow{\partial}_2 = (G^{-1})_{12} + q(\hat{\mathcal{R}}^{-1})_{12} \, \overleftarrow{\partial}_1 \, x_2 \\
&\eta_1 \, \overleftarrow{\partial}_2 = q^{-1}\hat{\mathcal{R}}_{12} \, \overleftarrow{\partial}_1 \, \eta_2 \\
&(\mathcal{P}_A)_{12} \, \overleftarrow{\partial}_1 \overleftarrow{\partial}_2 = 0
\end{aligned} \tag{22}$$

in which $\eta = (\eta^\mu)$, $\overleftarrow{\partial} = (\overleftarrow{\partial}^\mu)$. We call the right differential $\overleftarrow{\mathrm{d}}$ as dual of the previous one which is redenoted by left action,

$$\mathrm{d}f \equiv \overrightarrow{\mathrm{d}} \, f \tag{23}$$

and, a priori, there is no direct connction between η^μ and ξ^μ. If A' itself is still not consistent under $*$-conjugation, we now assume an extended algebra A'' generated by $\{x^\mu, \xi^\mu, \eta^\mu\}$ and graded by $\overrightarrow{\mathrm{d}}$ and $\overleftarrow{\mathrm{d}}$, with both differentials cooperating according to the rule

$$(\overrightarrow{\mathrm{d}} \, f) \, \overleftarrow{\mathrm{d}} = \overrightarrow{\mathrm{d}} \, (f \, \overleftarrow{\mathrm{d}}) = \overrightarrow{\mathrm{d}} \, f \, \overleftarrow{\mathrm{d}} = 0 \tag{24}$$

and with the $*$-conjugation by the relation

$$(\overrightarrow{\mathrm{d}} \, f)^* = (f^*) \, \overleftarrow{\mathrm{d}} \qquad (f \, \overleftarrow{\mathrm{d}})^* = \overrightarrow{\mathrm{d}} \, (f^*). \tag{25}$$

It is easy to see that the conjectures (4), (17), (24), (14), (25) are consistent, and the duality of differentials is, in some sense, $*$-conjugation in the algebra A''. From (24) we add to (8) and (22) the mixed commutations

$$\begin{aligned} &\xi_1\eta_2 = \eta_1\xi_2 \\ &(\mathcal{P}_S)_{12}\xi_1\eta_2 = (\mathcal{P}_1)_{12}\xi_1\eta_2 = 0 \\ &\vec{\partial}_1\overleftarrow{\partial}_2 = q^{-1}(\hat{\mathcal{R}}^{-1})_{12}\,\overleftarrow{\partial}_1\vec{\partial}_2 \\ &\vec{\partial}_1\,\eta_2 = q(\hat{\mathcal{R}}^{-1})_{12}\eta_1\,\vec{\partial}_2 \\ &\xi_1\,\overleftarrow{\partial}_2 = q(\hat{\mathcal{R}}^{-1})_{12}\,\overleftarrow{\partial}_1\,\xi_2 \end{aligned} \tag{26}$$

to complete the differential calculus in A''. The two sets of differentials ξ^μ and η^μ are conjugate with one another according to (15) and (25),

$$(\xi^\mu)^* = C^\mu{}_\nu\eta^\nu \qquad (\eta^\mu)^* = C^\mu{}_\nu\xi^\nu \tag{27}$$

and, similarly, so are $\overleftarrow{\partial}^\mu$ and $\vec{\partial}^\mu$,

$$(\vec{\partial}^\mu)^* = C^\mu{}_\nu\,\overleftarrow{\partial}^\nu \qquad (\overleftarrow{\partial}^\mu)^* = C^\mu{}_\nu\,\vec{\partial}^\nu\,. \tag{28}$$

By equations (15), (27), (28) and the fundamental rules (14) we can prove that the calculus defined by (8), (22), (26) is self-consistent under $*$-operation. Therefore, we see that introducing dual differentials is necessary. More generally this duality can be extended to differential calculus in any non-commutative C^*-algebra.

When $q \to 1$, this differential calculus approaches its classical limit, i.e. the x^μ commute with each other and with ξ^μ and η^μ, but ξ^μ and η^μ are anticommutative. In this case we can put

$$\eta^\mu = \xi^\mu \qquad (f\,\overleftarrow{\partial}^\mu) = (\vec{\partial}^\mu\,f) \tag{29}$$

without contradicting the above relations and, in fact, this means that

$$(\mathrm{d}f)^* = (-1)^{p(f)}\mathrm{d}f^*. \tag{30}$$

Therefore we recover the usual Cartan differential calculus compatible with $*$-conjugation in commutative geometry.

Remark. Relations of the type $\xi^\mu = \tilde{C}^\mu{}_\nu\xi^\nu$ with $\tilde{C}$ different from C were also suggested to avoid the extension of A to A''. The difficulty here is to find a suitable matrix $\tilde{C}$, which is invariant under quantum Lorentz transformation and make all the commutation relations in A (equation (8)) self-consistent under $*$-conjugation, especially those for the mixed products of x^μ, ξ^μ and ∂^μ, because this requires a series of compatibility conditions for matrices C, $\tilde{C}$ and $\hat{\mathcal{R}}$, and I failed to find such a matrix $\tilde{C}$.

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